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ON THE IRREDUCIBILITY OF DELIGNE-LUSZTIG VARIETIES

CÉDRIC BONNAFÉ & RAPHAËL ROUQUIER

ABSTRACT. Let \mathbf{G} be a connected reductive algebraic group defined over an algebraic closure of a finite field and let $F : \mathbf{G} \rightarrow \mathbf{G}$ be an endomorphism such that F^δ is a Frobenius endomorphism for some $\delta \geq 1$. Let \mathbf{P} be a parabolic subgroup of \mathbf{G} . We prove that the Deligne-Lusztig variety $\{g\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P} \cdot F(\mathbf{P})\}$ is irreducible if and only if \mathbf{P} is not contained in a proper F -stable parabolic subgroup of \mathbf{G} .

Let \mathbf{G} be a connected reductive group over an algebraic closure of a finite field and let $F : \mathbf{G} \rightarrow \mathbf{G}$ be an endomorphism such that some power of F is a Frobenius endomorphism of \mathbf{G} . If \mathbf{P} is a parabolic subgroup of \mathbf{G} , we set

$$\mathbf{X}_{\mathbf{P}} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P} \cdot F(\mathbf{P})\}.$$

This is the Deligne-Lusztig variety associated to \mathbf{P} . The aim of this note is to prove the following result:

Theorem A. *Let \mathbf{P} be a parabolic subgroup of \mathbf{G} . Then $\mathbf{X}_{\mathbf{P}}$ is irreducible if and only if \mathbf{P} is not contained in a proper F -stable parabolic subgroup of \mathbf{G} .*

Note that this result has been obtained independently by Lusztig (unpublished) and Digne and Michel [DiMi2, Proposition 8.4] in the case where \mathbf{P} is a Borel subgroup: both proofs are obtained by counting rational points of $\mathbf{X}_{\mathbf{P}}$ in terms of the Hecke algebra. We present here a geometric proof (inspired by an argument of Deligne [Lu, proof of Proposition 4.8]) which reduces the problem to the irreducibility of the Deligne-Lusztig variety associated to a Coxeter element: this case has been treated by Deligne and Lusztig [Lu, Proposition 4.8].

Before starting the proof of this Theorem, we first describe an equivalent statement. Let \mathbf{B} be an F -stable Borel subgroup of \mathbf{G} , let \mathbf{T} be an F -stable maximal torus of \mathbf{B} , let W be the Weyl group of \mathbf{G} relative to \mathbf{T} and let S be the set of simple reflections of W with respect to \mathbf{B} . We denote again by F the automorphism of W induced by F . Given $I \subset S$, let W_I denote the standard parabolic subgroup of W generated by I and let $\mathbf{P}_I = \mathbf{B}W_I\mathbf{B}$. We denote by \mathcal{P}_I the variety of parabolic subgroups of \mathbf{G} of type I (i.e. conjugate to \mathbf{P}_I) and by \mathcal{B} the variety of Borel subgroups of \mathbf{G} (i.e. $\mathcal{B} = \mathcal{P}_{\emptyset}$). For $w \in W$, we denote by $\mathcal{O}_I(w)$ the \mathbf{G} -orbit of $(\mathbf{P}_I, {}^w\mathbf{P}_{F(I)})$ in $\mathcal{P}_I \times \mathcal{P}_{F(I)}$. Note that $\mathcal{O}_I(w)$ depends only on the double coset $W_I w W_{F(I)}$. We define now

$$\mathbf{X}_I(w) = \{\mathbf{P} \in \mathcal{P}_I \mid (\mathbf{P}, F(\mathbf{P})) \in \mathcal{O}_I(w)\}.$$

The group \mathbf{G}^F acts on $\mathbf{X}_I(w)$ by conjugation. We set $\mathcal{O}(w) = \mathcal{O}_{\emptyset}(w)$ and $\mathbf{X}(w) = \mathbf{X}_{\emptyset}(w)$.

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Theorem A'. *Let $I \subset S$ and let $w \in W$. Then $\mathbf{X}_I(w)$ is irreducible if and only if $W_I w$ is not contained in a proper F -stable standard parabolic subgroup of W .*

REMARK 1 - Let us explain why the Theorems A and A' are equivalent. Let \mathbf{P}_0 be a parabolic subgroup of \mathbf{G} . Let I be its type and let $g_0 \in \mathbf{G}$ be such that $\mathbf{P}_0 = {}^{g_0}\mathbf{P}_I$. Let $w \in W$ be such that $g_0^{-1}F(g_0) \in \mathbf{P}_{Iw}\mathbf{P}_{F(I)}$. The pair $(I, W_I w W_{F(I)})$ is uniquely determined by \mathbf{P}_0 . Then, the map $\mathbf{X}_{\mathbf{P}_0} \rightarrow \mathbf{X}_I(w)$, $g\mathbf{P}_0 \mapsto {}^{gg_0}\mathbf{P}_I$ is an isomorphism of varieties (indeed, it is straightforward that $g^{-1}F(g) \in \mathbf{P}_0 \cdot F(\mathbf{P}_0)$ if and only if $(gg_0)^{-1}F(gg_0) \in \mathbf{P}_{Iw}\mathbf{P}_{F(I)}$).

Let \mathbf{Q} be a parabolic subgroup of \mathbf{G} containing \mathbf{P} . Let J be its type. Then $I \subset J$, $\mathbf{Q} = {}^{g_0}\mathbf{P}_J$ and $g_0^{-1}F(g_0) \in \mathbf{P}_{Jw}\mathbf{P}_{F(J)}$. Now, \mathbf{Q} is F -stable if and only if $F(J) = J$ and $w \in W_J$. This shows the equivalence of the two Theorems.

REMARK 2 - The condition “ $W_I w$ is not contained in a proper F -stable standard parabolic subgroup of W ” is equivalent to “ $W_I w W_{F(I)}$ is not contained in a proper F -stable standard parabolic subgroup of W ”.

The rest of this paper is devoted to the proof of Theorem A'. We fix a subset I of S and an element w of W . We first recall two elementary facts. If $I \subset J$, let $\tau_{IJ} : \mathcal{P}_I \rightarrow \mathcal{P}_J$ be the morphism of varieties that sends $\mathbf{P} \in \mathcal{P}_I$ to the unique parabolic subgroup of type J containing \mathbf{P} . It is surjective. Moreover,

$$(1) \quad \tau_{IJ}(\mathbf{X}_I(w)) \subset \mathbf{X}_J(w)$$

and

$$(2) \quad \tau_{IJ}^{-1}(\mathbf{X}_J(w)) = \bigcup_{W_I x W_{F(I)} \subset W_J w W_{F(J)}} \mathbf{X}_I(x).$$

First step: the “only if” part. Assume that there exists a proper F -stable subset J of S such that $W_I w \subset W_J$. Then, by (1), we have $\tau_{IJ}(\mathbf{X}_I(w)) \subset \mathbf{X}_J(1) = \mathcal{P}_J^F$. Since \mathbf{G}^F acts transitively on \mathcal{P}_J^F , we get $\tau_{IJ}(\mathbf{X}_I(w)) = \mathbf{X}_J(1)$. This shows that $\mathbf{X}_I(w)$ is not irreducible.

Second step: reduction to Borel subgroups. By the previous step, we can concentrate on the “if” part. So, from now on, we assume that $W_I w$ is not contained in a proper F -stable parabolic subgroup of W . Then, by (2), we have

$$\tau_{\emptyset I}^{-1}(\mathbf{X}_I(w)) = \bigcup_{x \in W_I w W_{F(I)}} \mathbf{X}(x).$$

Let v denote the longest element of $W_I w W_{F(I)}$. Then every element x of the double coset $W_I w W_{F(I)}$ satisfies $x \leq v$ (here, \leq denotes the Bruhat order on W): this follows for instance from the fact that $\mathbf{P}_{Iw}\mathbf{P}_{F(I)}$ is irreducible and is equal to $\bigcup_{x \in W_I w W_{F(I)}} \mathbf{B}w\mathbf{B}$. In particular, v is not contained in a proper F -stable parabolic subgroup of W .

Now, let $\mathbf{X}' = \bigcup_{x \in W_I w W_{F(I)}} \mathbf{X}(x)$. Then, since $\overline{\mathbf{X}(v)} = \bigcup_{x \leq v} \mathbf{X}(x)$, we have

$$\mathbf{X}(v) \subset \mathbf{X}' \subset \overline{\mathbf{X}(v)}.$$

So, since $\tau_{\emptyset I}(\mathbf{X}') = \mathbf{X}_I(w)$, it is enough to show that $\mathbf{X}(v)$ is irreducible. In other words, we may, and we will, assume that $I = \emptyset$.

Third step: smooth compactification. Let (s_1, \dots, s_n) be a finite sequence of elements of S . Let

$$\begin{aligned} \hat{\mathbf{X}}(s_1, \dots, s_n) = \{(\mathbf{B}_1, \dots, \mathbf{B}_n) \in \mathcal{B}^n \mid (\mathbf{B}_n, F(\mathbf{B}_1)) \in \overline{\mathcal{O}(s_n)} \\ \text{and } (\mathbf{B}_i, \mathbf{B}_{i+1}) \in \overline{\mathcal{O}(s_i)} \text{ for } 1 \leq i \leq n-1\}. \end{aligned}$$

If $\ell(s_1 \cdots s_n) = n$, then $\hat{\mathbf{X}}(s_1, \dots, s_n)$ is a smooth compactification of $\mathbf{X}(s_1 \cdots s_n)$ (see [DeLu, Lemma 9.11]): in this case,

(3) $\mathbf{X}(s_1 \cdots s_n)$ is irreducible if and only if $\hat{\mathbf{X}}(s_1, \dots, s_n)$ is irreducible.

Note that $(\mathbf{B}, \dots, \mathbf{B}) \in \hat{\mathbf{X}}(s_1, \dots, s_n)$. We denote by $\hat{\mathbf{X}}^\circ(s_1, \dots, s_n)$ the connected (i.e. irreducible) component of $\hat{\mathbf{X}}(s_1, \dots, s_n)$ containing $(\mathbf{B}, \dots, \mathbf{B})$. Let $H(s_1, \dots, s_n) \subset \mathbf{G}^F$ be the stabilizer of $\hat{\mathbf{X}}^\circ(s_1, \dots, s_n)$. Let us now prove the following fact:

(4) if $1 \leq i_1 < \dots < i_r \leq n$, then $H(s_{i_1}, \dots, s_{i_r}) \subset H(s_1, \dots, s_n)$.

Proof of (4) - The map $f : \hat{\mathbf{X}}(s_{i_1}, \dots, s_{i_r}) \longrightarrow \hat{\mathbf{X}}(s_1, \dots, s_n)$ defined by

$$f(\mathbf{B}_1, \dots, \mathbf{B}_1) = (\mathbf{B}_1, \dots, \underbrace{\mathbf{B}_1}_{i_1\text{-th position}}, \mathbf{B}_2, \dots, \underbrace{\mathbf{B}_{r-1}}_{i_{r-1}\text{-th position}}, \mathbf{B}_r, \dots, \underbrace{\mathbf{B}_r}_{i_r\text{-th position}}, F(\mathbf{B}_1), \dots, F(\mathbf{B}_1))$$

is a \mathbf{G}^F -equivariant morphism of varieties. Moreover,

$$f(\underbrace{(\mathbf{B}, \dots, \mathbf{B})}_{r \text{ times}}) = \underbrace{(\mathbf{B}, \dots, \mathbf{B})}_{n \text{ times}}.$$

In particular, $f(\hat{\mathbf{X}}^\circ(s_{i_1}, \dots, s_{i_r}))$ is contained in $\hat{\mathbf{X}}^\circ(s_1, \dots, s_n)$. This proves the expected inclusion between stabilizers. ■

Last step: twisted Coxeter element. The quotient variety

$$\mathbf{G}^F \backslash \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{B}w\mathbf{B}\}$$

is irreducible (it is isomorphic to $\mathbf{B}w\mathbf{B}$ through the Lang map $\mathbf{G}^F g \mapsto g^{-1}F(g)$), hence $\mathbf{G}^F \backslash \mathbf{X}(w)$ is irreducible as well. So,

(5) \mathbf{G}^F permutes transitively the irreducible components of $\mathbf{X}(w)$.

Let $w = s_1 \cdots s_n$ be a reduced decomposition of W as a product of elements of S . By (3) and (5), it suffices to show that $H(s_1, \dots, s_n) = \mathbf{G}^F$. Since w does not belong to any F -stable proper parabolic subgroup of W , there exists a sequence $1 \leq i_1 < \dots < i_r \leq n$ such that $(s_{i_k})_{1 \leq k \leq r}$ is a family of representatives of F -orbits in S . By (4), we have $H(s_{i_1}, \dots, s_{i_r}) \subset H(s_1, \dots, s_n)$. But, by [Lu, Proposition 4.8], $\mathbf{X}(s_{i_1} \cdots s_{i_r})$ is irreducible so, again by (3) and (5), $H(s_{i_1}, \dots, s_{i_r}) = \mathbf{G}^F$. Therefore, $H(s_1, \dots, s_n) = \mathbf{G}^F$, as expected.

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